Trig Equations

Review

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note for below: $90° = \frac{\pi}{2}$

$$\sin\theta = \cos(90° - \theta) \qquad \csc\theta = \sec(90° - \theta) \qquad \tan\theta = \cot(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta) \qquad \sec\theta = \csc(90° - \theta) \qquad \cot\theta = \tan(90° - \theta)$$

Trig Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = \tan^2\theta + 1 \qquad \csc^2\theta = \cot^2\theta + 1$$

Sum and Difference Equations

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Double Angles

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Polar Graphs

Converting Equations

$$r^2 = x^2 + y^2$$

$$\sin\theta = \frac{y}{r} \to y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \to x = r\cos\theta$$

$$\tan\theta = \frac{y}{x} \to \theta = \tan^{-1}\frac{y}{x}$$

Polar coordinates

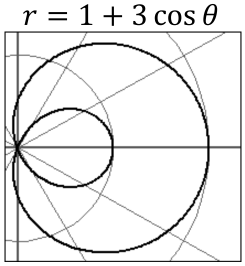
Always given as (r, θ)

Symmetry

- If the coefficient on the sine or cosine function is negative then most of the graph will be in II and III or III and IV quadrants respectively

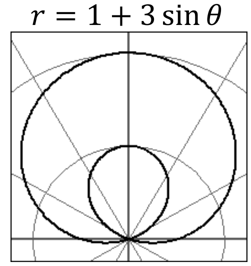
If cosine then x-axis

$r = 1 + 3\cos\theta$



If sine then y-axis

$r = 1 + 3\sin\theta$



The 3 Forms

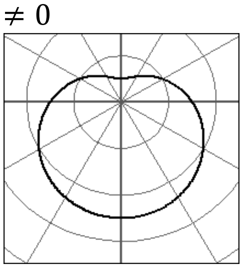
Limaçons

$r = a \pm b\sin\theta$ or $a \pm b\sin\theta$ $a \neq 0$

- Intercepts on <u>non-symmetry</u> axis are $\pm a$
- Find other intercepts by substituting values for θ on the <u>symmetry</u> axis which make the trig function ± 1 and solve

Dimpled

Hint: $r \neq 0$

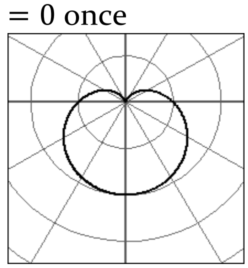


$r = 3 - 2\sin\theta$

$|a| > |b|$

Cardioid

Hint: $r = 0$ once

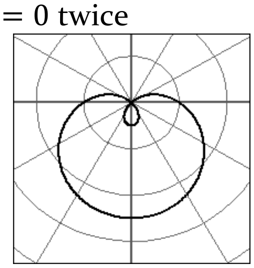


$r = 2 - 2\sin\theta$

$|a| = |b|$

Loop

Hint $r = 0$ twice



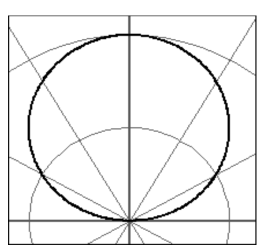
$r = 2 - 3\sin\theta$

$|a| < |b|$

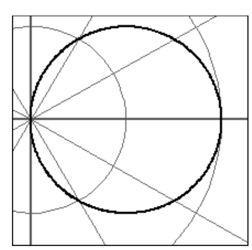
Circles

$r = b\sin\theta$ or $r = b\cos\theta$

- Diameter of circle is $|b|$
- If **Sine** then circle is centered on **y-axis**
- If **Cosine** then circle is centered on **x-axis**
- If $b < 0$ (aka negative) then circle **flips** over **non-symmetrical axis**



$r = 4\sin\theta$

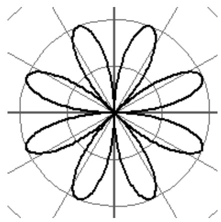


$r = 4\cos\theta$

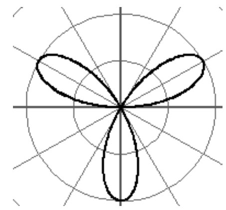
Roses

$r = \pm a \sin n\theta$ or $r = \pm a \cos n\theta$

- If n is **even** then $2n$ petals
- If n is **odd** then n petals
- **Cosine begins** on polar axis; **sine begins** at the θ when you solve $a = a \sin n\theta$
- To find **distance between petals** do $\frac{2\pi}{\text{\# of petals}}$ or $\frac{360°}{\text{\# of petals}}$ because the petals are evenly spaced



$r = 4 \sin 4\theta$



$r = 4 \sin 3\theta$

Solving systems of polar equations

1. Solve for theta over 2π to find simultaneous solutions
2. Graph and find the non-simultaneous solutions (Hint: look for symmetry)
3. Don't forget how to solve Trig. Equations & the identities

Polar Complex Numbers

- $(a + bi)$ is graphed at the point (a, b) in rectangular form
- In polar form it is $(r, \theta) = (\sqrt{a^2 + b^2}, \tan^{-1} \frac{b}{a})$
- To convert either use above equation or plot (a, b) on an axis and draw a triangle to find r & θ
- **Absolute value** (aka **modulus**) is the distance to origin: $|a + bi| = \sqrt{a^2 + b^2}$
- $z = r\ cis\theta$ is shorthand for $z = r(\cos\theta + i \sin\theta)$
- Multiplication: $(r\ \text{cis}A) \cdot (s\ \text{cis}B) = rs\ \text{cis}(A + B)$
- Division: $\frac{(r\ \text{cis}\,A)}{(s\ \text{cis}\,B)} = \frac{r}{s}\ \text{cis}(A - B)$
- de Moivre's Theorem: $z^n = r^n \text{cis}(n\theta)$
  - Used to find squares & cube roots etc.
  - To find all solutions add $\frac{2\pi}{number\ \ of\ solutions}$ to θ in $r^n \text{cis}(n\theta)$ and solve

If you don't know what this chart is then just ignore it…

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	1	0	—	1	—
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	—	1	—	0

Conics Review Sheet

General

Graphing parts of a conic

- **Top half** – solve for **y** and take **positive** square root
- **Bottom half** – solve for **y** and take **negative** square root
- **Right half** – solve for **x** and take **positive** square root
- **Left half** – solve for **x** and take **negative** square root

Circles

Graphing form: $(x-h)^2+(y-k)^2=r^2$

Ellipses

Graphing form

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

- Note: Don't get too hung up on what a is and what b is, just remember to put the direction from the center in the **x-direction under the x** and the **y-direction distance under the y**.
- **Eccentricity**: $e=\frac{c}{a}$
  - If e is close to 0 then the ellipse will be more circular, if close to 1 then stretched out
- $c^2=a^2-b^2$
  - Make sure a is the **larger number** regardless of if it is the x or y distance

Hyperbolas

Graphing form

Opens in x-direction (left & right), x is positive

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Opens in y-direction (up & down), y is positive

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

- $c^2=a^2+b^2$
- **Foci** always located in the opening of the hyperbola
- Eccentricity is the same as ellipse (e is always greater than 1)
- Asymptotes are found by making a rectangle connecting the four corners $(\pm a, \pm b)$ and drawing diagonal lines through the diagonally opposite corners

Parabolas

- Use **vertex form**:

$$y=a(x-h)^2+k$$

Opens in y-direction (up or down)

$$x=a(y-k)^2+h$$

Opens in x-direction (left or right)

- **Focal diameter** – distance parallel to directrix between 2 points on parabola through the focus
- **Focal Point** – distance of p from the vertex in the direction the parabola opens
- $a=\frac{1}{4p}=\frac{1}{\text{focal diameter}}$
  - focal diameter $=4p$
- Distance from p to Directrix is $|y+p|$
- **Directrix** – line that is a distance of p from the vertex in the opposite direction as the focus; it is parallel to the tangent line of the vertex
  - Generally it will be $x=\pm p$ or $y=\pm p$ (note: $\pm$ means plus **OR** minus not both, there is **only one directrix**)

Sequences and Series

- Sequence – a list of numbers
- Series – a list of numbers with + between them (a sum)

Arithmetic Sequences - a list of numbers with a common difference (it is linear)

- **Common Difference Formula**:

$$a_n = dn + a_o$$

- Also, use coordinate points to find the a_n formula ex. $a_5 = 34 \rightarrow (5,34)$ or (n, a_n)
    - d is equivalent to the slope so,

$$d = \frac{a_{n1} - a_{n2}}{n_1 - n_2}$$

    - Solve using 2 "pairs" like for an equation of a line ($y = mx + b \Rightarrow a_n = dn + a_0$)
- **Explicit Rule** – a rule that can be used to find any term using "n"
- **Recursive rule** – A rule that uses the term before the one you are finding
- **Fibonacci sequence**: $a_n = a_{n-1} + a_{n-2}$ where $a_1 = 1 \; and \; a_2 = 1$
    - 1,1,2,3,5,8,13,21 …
- **Partial Sum** is a sum of part of a sequence
    - If given the first, last, and number of pairs then

$$s_n = \frac{\text{\# of terms}}{2}(1^{\text{st}} \text{ term} + \text{last term})$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Geometric Sequences – a list of numbers with a common multiplier

$$a_n = a_1 r^{n-1}$$

$$r = \frac{a_t}{a_{t-1}} = \frac{\text{term}}{\text{previous term}}$$

- **Verify that the sequence is geometric** by finding the r value at least twice

$$s_n = \frac{a_1(1 - r^n)}{1 - r}$$

- Infinite geometric sequence
    - When $|r| \geq 1$ the sequence **diverges and there is <u>no ∞ sum</u>**
        - Note: if r = 1 then not really geomentric
    - when $|r| < 1$ the sequence **converges**

$$s_\infty = \frac{a_1}{1 - r}$$

Quadratic Sequences

- the second differences (the difference of the differences) show the coefficient on the n^2 value
- the 2nd difference is twice the coefficient (if 2nd diff. = 2 then n^2 if 4 then $2n^2$)
- To find a_n rule find n^2 and subtract it from the sequence
- Then write a rule for the remaining values (as a normal geometric sequence)

Intervals of Convergence

- Find r and set the absolute value of r less than one $\quad |r| < 1$
- Solve the inequality
- Note: if x is in the denominator then set the absolute value of it greater than the absolute value of the numerator
    - Ex. $r = \left|-\frac{3}{x+4}\right| \;\Rightarrow\; \left|-\frac{3}{x+4}\right| < 1 \;\Rightarrow\; |x+4| > 3$
- Solve for values of x, these values are the intervals of convergence

Infinite Repeating Decimals

1. Find the repeating part as a fraction
    - Ex: $.3\overline{15} \implies .3 + .015 + .00015 \ldots \implies \frac{3}{10} + \frac{15}{1000} + \frac{15}{100000} \ldots$
2. Find r ignoring the non-repeating part (ex. In above sample $r = \frac{1}{100}$)
3. Solve infinite geometric series using a_1 and r; this gives you the repeating part as a fraction
    - Ex: $s_\infty = \frac{a_1}{1-r} \implies \frac{\frac{15}{1000}}{1-\frac{1}{100}} \implies \frac{5}{330}$
4. Add the non-repeating part
    - Ex: $\frac{5}{330} + \frac{3}{10} = \frac{104}{330}$

Math Induction

Limit Intro

- Find like horizontal asymptote for rational function

$$\lim_{n\_\to\infty} n^2 = DNE$$

$$\lim_{n\to\infty} \frac{3n+2}{2n} = \frac{3}{2}$$

Math induction proof for sum of a series

1. Show that the rule holds true for some value of n, usually it's best to try when $n = 1$
2. Assume the rule works $n = k$ and show that it also works for the following term, $k + 1$
    a. Hint: remember that you assumed that some part of the sum was true, substitute this value now
3. Show that when you add the a_{k+1} term is equal to S_{k+1}.
    a. Make sure to only work one side at a time

Math induction proof for divisibility of an expression (ex. Prove $n^2 + 5n$ is divisible by 2)

1. Show that the rule holds true for some value of n, usually it's best to try when $n = 1$
2. Assume the rule works $n = k$ and show that it also works for the term $k + 1$
    a. Hint: remember that your assumed value and all multiples are divisible
    b. Hint: Remember that $5^{k+1} = 5^1 5^k$
    c. Hint: remember that $5^1 5^k = 5^k(3 + 2)$ so you can distribute the 5^k and possibly factor to get your assumed value
3. Show that when you add the a_{k+1} term is equal to S_{k+1}.
    a. Make sure to only work one side at a time

Binomial Theorem - Shows how to expand $(\boldsymbol{x} + \boldsymbol{y})^{\boldsymbol{n}}$

- Things to remember:
    - Number of terms is $n + 1$
    - 1st term is x^n and last is y^n
    - Exponents on x **decrease** and on y the **increase**(the variables x & y may differ)
    - In each term the exponents must add to n
    - Coefficients follow Pascal's Triangle
    - The n^{th} coefficient can be found by $\frac{n!}{(exponent\ \ on\ x)!\cdot(exponent\ \ on\ y)!}$
    - Don't forget to consider the negative terms in $(x - y)^n$

$$(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$$

Pascal's Triangle (1st 3 rows, the row with 1 in it is the 0th row)

1

1 1

1 2 1

1 3 3 1

Limits

Note: for this review sheet $\lim_{x \to a} f(x)$ is the same as $\lim\limits_{x \to a} f(x)$

Introduction

- $\lim\limits_{x \to a^-} f(x)$: the limit as x approaches "a" from the left
- $\lim\limits_{x \to a^+} f(x)$: the limit as x approaches "a" from the right
- "**The big limit**" – If the limit equals the limit from the left, and the value is a real, finite number, then the limit exists
- **Limit exists** if the function is continuous or has a hole at "a"
- **Limit does not exist (DNE)** if the function has a V.A., a jump, or an endpoint
    - If either of the limits from the left or right are ∞ then the limit DNE

Laws of Limits

Suppose: $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ then:

$$\lim_{x \to a}\big(f(x) \pm g(x)\big) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\big(f(x) * g(x)\big) = \lim_{x \to a} f(x) * \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

$$\lim_{x \to a} c * f(x) = c * \lim_{x \to a} f(x)$$

$$\lim_{x \to a}\big(f(x)\big)^n = \left(\lim_{x \to a} f(x)\right)^n$$

Steps to solving limits (when limit is found then stop)

1. Evaluate function at $x = a$ if real finite number then the limit is found (make sure to check that limit is same from left a^- as it is from the right a^+)
    a. If you get $\frac{0}{0}$ then it is a hole
2. See if the function reduces by adding fractions or factoring etc. then re-evaluate (go to step 1)
3. If there is a radical, multiply by the conjugate to rationalize numerator/denominator then evaluate (go to step 1)

- Determine horizontal asymptote for $\lim_{x \to \infty} f(x)$
    - No limit for slant asymptote (top heavy)
    - If powers equal then use coefficients
    - If bottom heavy then $\lim_{x \to \infty} f(x) = 0$
- Be specific when you write the graphical significance, for example Odd V.A. at $x = 2$

Definitions

- Continuity – a function has no jumps, holes (removable discontinuities), etc.
    - $f(x)$ is continuous at $x = c$ if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$ note: $f(c)$ must exist
- Removable discontinuity (hole) – a point in the function where x is undefined